2 Frequency-Domain Analysis

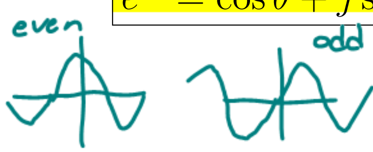
Electrical engineers live in the two worlds, so to speak, of time and frequency. Frequency-domain analysis is an extremely valuable tool to the communications engineer, more so perhaps than to other systems analysts. Since the communications engineer is concerned primarily with signal bandwidths and signal locations in the frequency domain, rather than with transient analysis, the essentially steady-state approach of the (complex exponential) **Fourier series** and **transforms** is used rather than the Laplace transform.

2.1 Mathematical Background

2.1. ***Euler's formula***:

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta.} \qquad \theta = 2\pi f_0 t \tag{1}$$

$\to e^{j2\pi f_0 t}$



$$e^{j(-\theta)} = \cos(-\theta) + j\sin(-\theta) = \cos(\theta) - j\sin(\theta)$$

when $f_0 > 0$, $e^{j2\pi f_0 t}$ rotates CCW

< 0, CW

$\times \frac{1}{2}$

$$\cos(x) = \text{Re}\left\{e^{jx}\right\} = \frac{1}{2}\left(e^{jx} + e^{-jx}\right) \tag{2}$$

$$\sin(x) = \text{Im}\left\{e^{jx}\right\} = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right). \tag{3}$$

2.2. We can use $\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$ and $\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$ to derive many trigonometric identities. See Example 2.4.

Example 2.3. Use the Euler's formula to show that $\frac{d}{dx}\sin x = \cos x$.

$$\frac{d}{dx}\sin x = \frac{d}{dx}\left(\frac{1}{2j}\left(e^{jx} - e^{-jx}\right)\right) = \frac{1}{2\cancel{j}}\left(\cancel{j}e^{jx} - (\cancel{-j})e^{-jx}\right) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$$

Example 2.4. Use the Euler's formula to show that $\cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$.

$$\cos^2(x) = \Big(\frac{1}{2}(\underbrace{e^{jx}}_{A} + \underbrace{e^{-jx}}_{B})\Big)^2 = \frac{1}{4}\left((A+B)^2\right) = \frac{1}{4}\left(A^2 + 2AB + B^2\right)$$

$$= \frac{1}{4}\Big(e^{j2x} + 2\underbrace{e^{j0}}_{1} + e^{-j2x}\Big) = \frac{1}{4}\left(2 + 2\cos(2x)\right)$$

2.5. Similar technique gives

(a) $\cos(-x) = \cos(x)$,

(b) $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$,

(c) $\sin^2 x = \frac{1}{2}\left(1 - \cos\left(2x\right)\right)$

(d) $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, and

(e) the **product-to-sum formula**

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right). \tag{4}$$

2.2 Continuous-Time Fourier Transform

Definition 2.6. The (direct) **Fourier transform** of a signal $g(t)$ is defined by

spectrum of $g(t)$

$$G(f) = \int_{-\infty}^{+\infty} g(t) e^{-j2\pi f t} dt \tag{5}$$

$|G(f)|$: amplitude spectrum

$\angle G(f)$ or $\arg(G(f))$: phase spectrum

This provides the frequency-domain description of $g(t)$. Conversion back to the time domain is achieved via the **inverse (Fourier) transform**:

$$g\left(t\right) = \int_{-\infty}^{\infty} G\left(f\right) e^{j2\pi f t} df \tag{6}$$

- We may combine (5) and (6) into one compact formula:

$$\boxed{\int_{-\infty}^{\infty} G\left(f\right) e^{j2\pi f t} df = g\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G\left(f\right) = \int_{-\infty}^{\infty} g\left(t\right) e^{-j2\pi f t} dt.} \tag{7}$$

- We may simply write $G = \mathcal{F}\left\{g\right\}$ and $g = \mathcal{F}^{-1}\left\{G\right\}$.

- Note that the area under the curve of a function in one domain is the same as its value at 0 in another domain:

$$G(0) = \int_{-\infty}^{\infty} g(t) dt \quad \text{and} \quad g(0) = \int_{-\infty}^{\infty} G(f) df. \tag{8}$$

Time Domain — Frequency Domain

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \underset{\text{inverse transform}}{\overset{\text{direct transform}}{\overset{\mathcal{F}}{\rightleftharpoons}}} G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

Capital letter is used to represent corresponding signal in the frequency domain.

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \overset{\mathcal{F}}{\rightleftharpoons} G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

Complex exponential: $e^{j2\pi ft} = \underbrace{\cos(2\pi ft) + j\sin(2\pi ft)}_{\text{sinusoids}}$

The relationship on the left is simply a **decomposition** of the signal $g(t)$ into a **linear combination** of (potentially infinitely many) $e^{j2\pi ft}$ components at different frequencies.

$$g(t) = \int_{-\infty}^{\infty} \boxed{G(f)} e^{j2\pi ft} df \overset{\mathcal{F}}{\rightleftharpoons} \boxed{G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt}$$

From the decomposition point of view, the value of $G(f)$ at a particular frequency f is simply the **weight** (scaling/coefficient) which tells how much $e^{j2\pi ft}$ component there is in $g(t)$.

By the **orthogonality** among complex exponential functions, the value of $G(f)$ at a particular frequency f can be calculated by the formula above.

This coefficient $G(f)$ considered as a function of frequency is the **Fourier transform** of our signal.

Figure 2: Fourier coefficients from the decomposition in the time domain.

2.7. In some references[5], the (direct) Fourier transform of a signal $g(t)$ is defined by

$$\hat{G}(\omega) = \int_{-\infty}^{+\infty} g(t) e^{-j\omega t} dt \tag{9}$$

In which case, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(\omega) e^{j\omega t} d\omega = g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \tag{10}$$

- In `MATLAB`, these calculations are carried out via the commands `fourier` and `ifourier`.
- Note that $\hat{G}(0) = \int_{-\infty}^{\infty} g(t)dt$ and $g(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(\omega)d\omega$.
- The relationship between $G(f)$ in (5) and $\hat{G}(\omega)$ in (9) is given by

$$G(f) = \hat{G}(\omega)\Big|_{\omega = 2\pi f} \tag{11}$$

$$\hat{G}(\omega) = G(f)|_{f = \frac{\omega}{2\pi}} \tag{12}$$

Before we introduce our first but crucial transform pair in Example 2.13 which will involve rectangular function, we want to introduce the indicator function which gives compact representation of the rectangular function. We will see later that the transform of the rectangular function gives a sinc function. Therefore, we will also introduce the sinc function as well.

Definition 2.8. An **indicator function** gives only two values: 0 or 1. It is usually written in the form

Notation 1: $$1[\text{some condition(s) involving } t].$$

Its value at a particular t is one if and only if the condition(s) inside is satisfied for that t.

Ex. $1[3 < t \le 5] = \begin{cases} 1, & 3 < t \le 5, \\ 0, & \text{otherwise.} \end{cases}$

[5] `MATLAB` uses this definition.

Alternatively, we can use a set to specify the values of t at which the indicator function gives the value 1:

Notation 2:

$$1_A(t) = \begin{cases} 1, & t \in A, \\ 0, & t \notin A. \end{cases}$$

In particular, the set A can be some intervals:

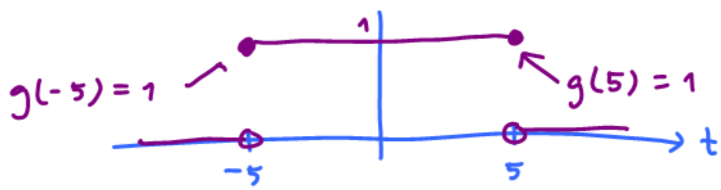
$$1_{[-a,a]}(t) = \begin{cases} 1, & -a \le t \le a, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$1_{[-a,b]}(t) = \begin{cases} 1, & -a \le t \le b, \\ 0, & \text{otherwise.} \end{cases}$$

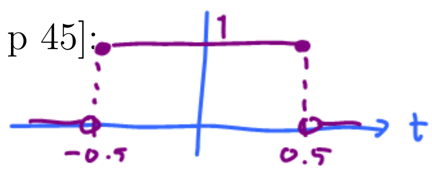
$-5 \le t \le 5$

Example 2.9. Carefully sketch the function $g(t) = 1\left[|t| \le 5\right]$ $= \begin{cases} 1, & |t| \le 5, \\ 0, & \text{otherwise} \end{cases}$

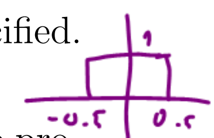


$= 1_{[-5,5]}(t)$

Definition 2.10. **Rectangular pulse** [3, Ex 2.21 p 45]:



- $\prod(t) = 1\left[|t| \le 0.5\right] = 1_{[-0.5,0.5]}(t)$
  - This is a pulse of unit height and unit width, centered at the origin. Hence, it is also known as the **unit gate** function rect (t) [5, p 78].
  - In [3], the values of the pulse $\prod(t)$ at -0.5 and 0.5 are not specified. However, in [5], these values are defined to be 0.5.



  - In MATLAB, the function `rectangularPulse(t)` can be used to produce[6] the unit gate function above. More generally, we can produce a rectangular pulse whose rising edge is at a and falling edge is at b via `rectangularPulse(a,b,t)`.
- $\prod\left(\frac{t}{T_0}\right) = 1\left[|t| \le \frac{T_0}{2}\right] = 1_{\left[-\frac{T_0}{2},\frac{T_0}{2}\right]}(t)$
  - Observe that T_0 is the width of the pulse.

[6]Note that `rectangularPulse(-0.5)` and `rectangularPulse(0.5)` give 0.5 in MATLAB.

Definition 2.11. The **sinc function**

$$\mathrm{sinc}(x) \equiv (\sin x)/x \tag{13}$$

is plotted in Figure 3.

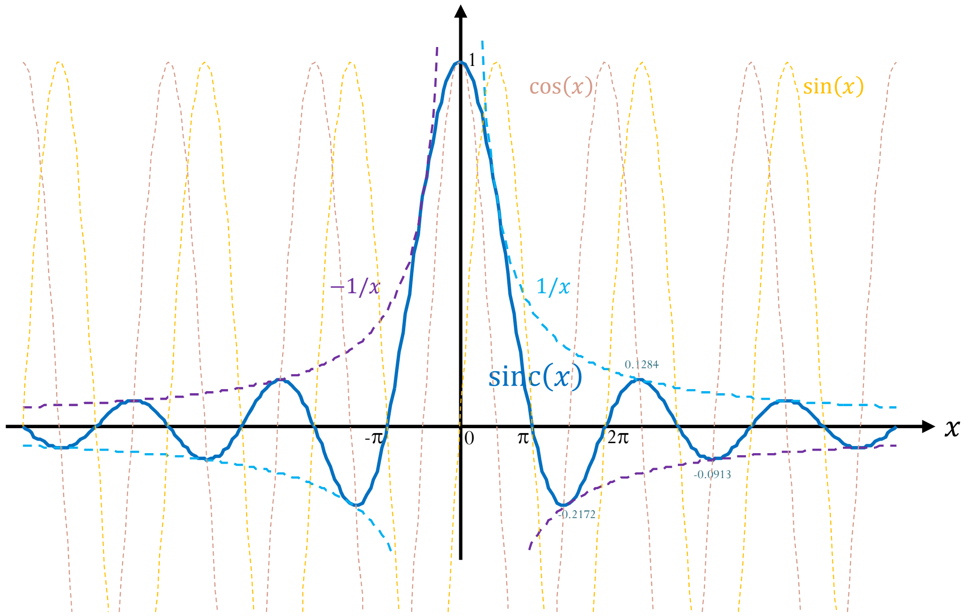


Figure 3: Sinc function

- This function plays an important role in signal processing. It is also known as the filtering or interpolating function.
    - The full name of the function is "sine cardinal"[7].
- Using L'Hôpital's rule, we find $\lim_{x \to 0} \mathrm{sinc}(x) = 1$.
- $\mathrm{sinc}(x)$ is the product of an oscillating signal $\sin(x)$ (of period 2π) and a monotonically decreasing function $1/x$. Therefore, $\mathrm{sinc}(x)$ exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $1/x$.
- Its zero crossings are at all non-zero integer multiples of π.

[7]which corresponds to the Latin name sinus cardinalis. It was introduced by Woodward in his 1952 paper "Information theory and inverse probability in telecommunication" [12], in which he noted that it "occurs so often in Fourier analysis and its applications that it does seem to merit some notation of its own"

Definition 2.12. Normalized sinc function:

In MATLAB and in many standard textbooks such as [3, p 37], [14, eq. 2.64], and [12], the function $\operatorname{sinc}(x)$ is defined as

$$\frac{\sin(\pi x)}{\pi x}. \tag{14}$$

- Its zero crossings are at non-zero integer values of its argument as shown in Figure 4.

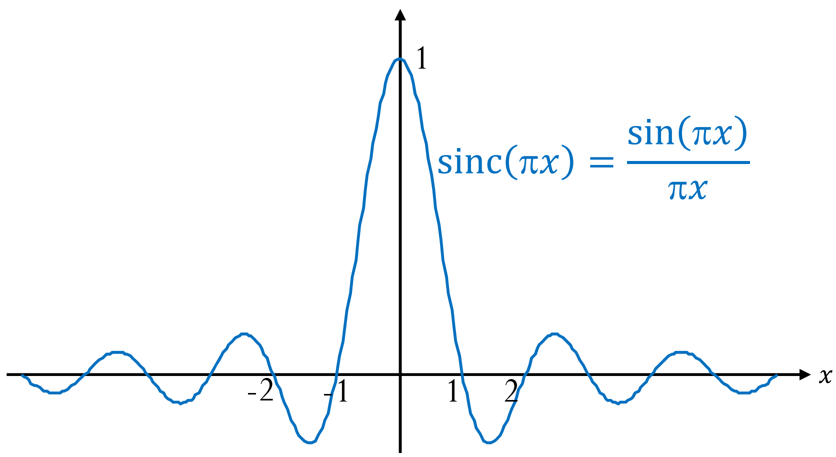


Figure 4: Normalized sinc function

- The "*normalized*" part of the name is added to distinguish it from (13) which is unnormalized .

2.13. Rectangular function and sinc function as a Fourier transform pair:

$$1\left[|t| \le a\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{\sin(2\pi f a)}{\pi f} = 2a \operatorname{sinc}(2\pi f a) \tag{15}$$

The right half of Figure 5 illustrates (15). By setting $a = T_0/2$, we have $1\left[|t| \le \frac{T_0}{2}\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} T_0 \operatorname{sinc}(\pi T_0 f)$. In particular, when $T_0 = 1$, we have

$$\operatorname{rect}(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \operatorname{sinc}(\pi f).$$

The Fourier transform of the unit gate function is the normalized sinc function.

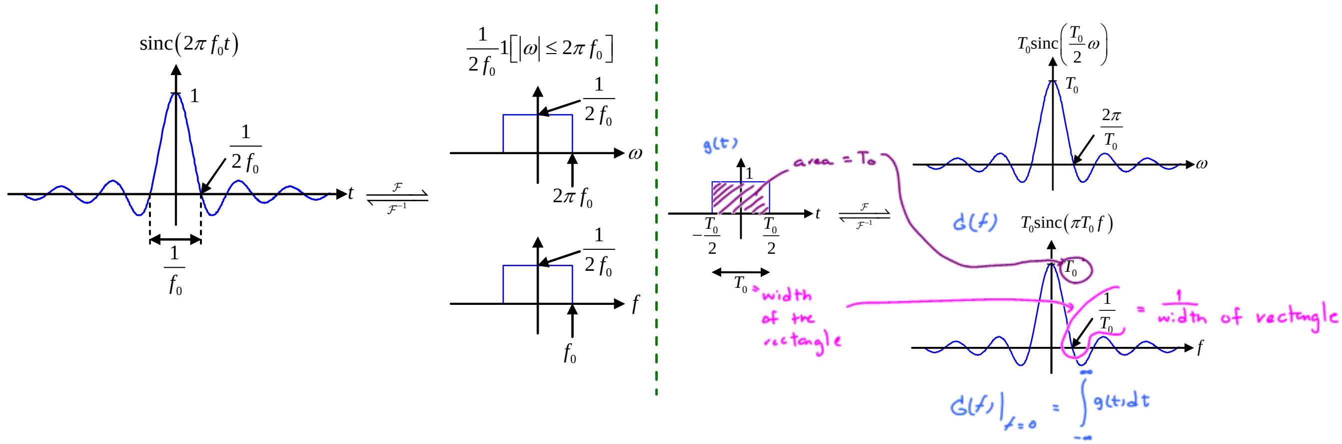


Figure 5: Fourier transform of sinc and rectangular functions

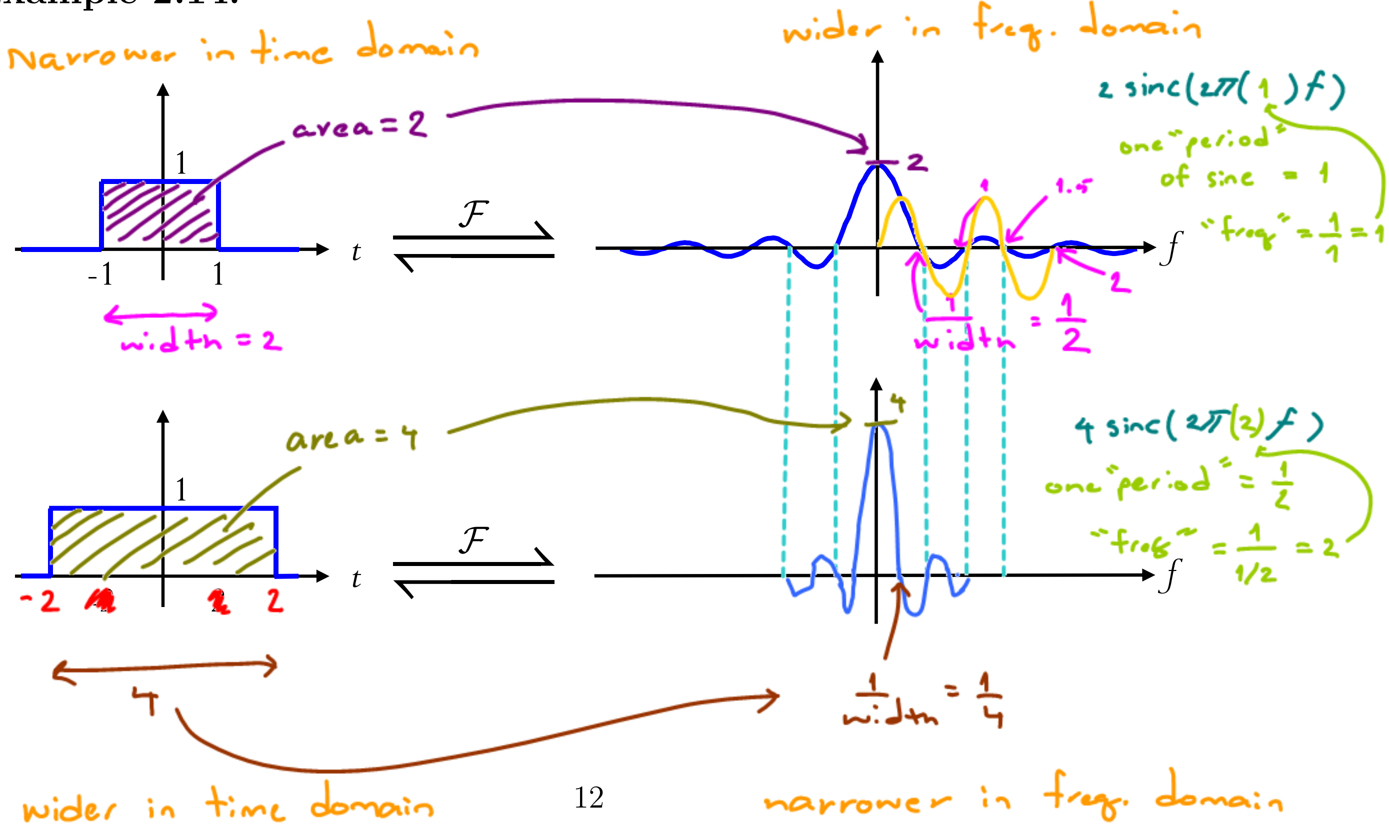
Observe that

(a) The height of the sinc function's peak is the same as the area under the rectangular function.

- This follows from (8).

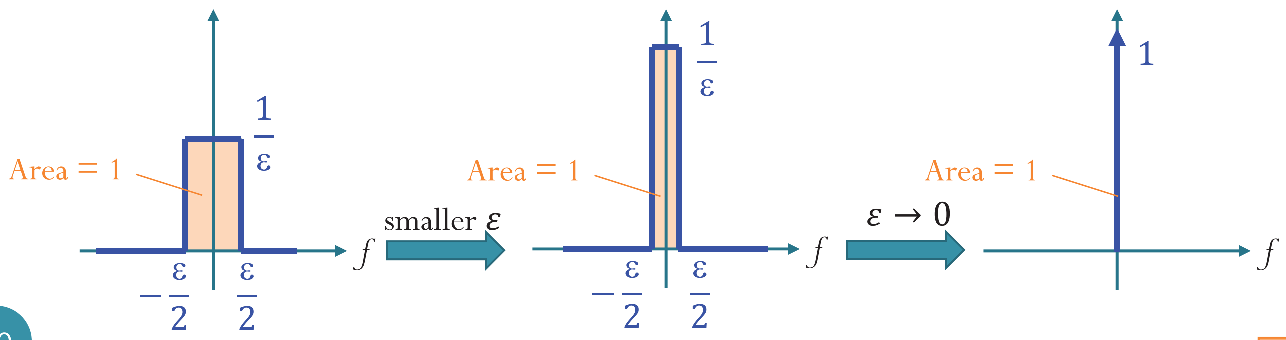
(b) The first zero crossing of the sinc function occurs at 1/(width of the rectangular function).

Example 2.14.



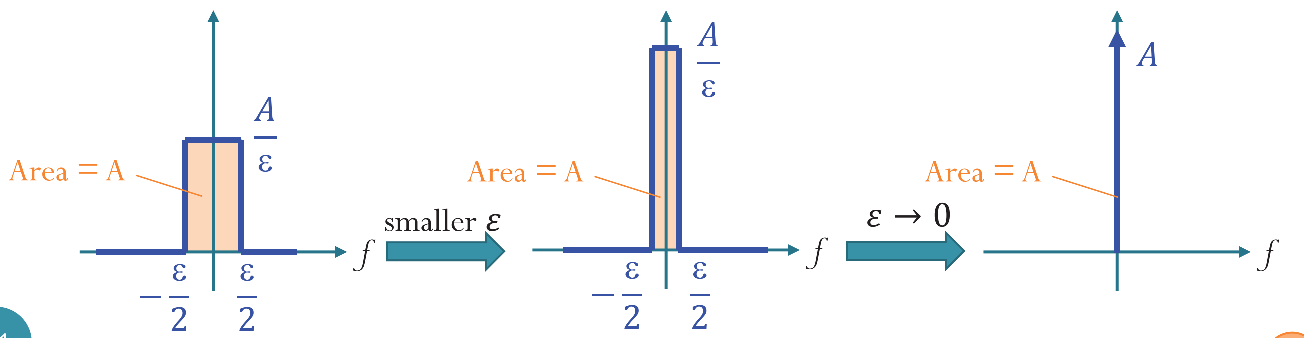
Delta function δ(*f*) [Definition 2.15]

- (Dirac) delta function or (unit) impulse function
- Usually depicted as a vertical arrow at the origin
- Not a true function
  - Undefined at $f = 0$
- Intuitively we may visualize $\delta(f)$ as an infinitely tall, infinitely narrow rectangular pulse of **unit area**



*A*δ(*f*)

- (Dirac) delta function or (unit) impulse function
- Usually depicted as a vertical arrow at the origin
- Not a true function
  - Undefined at $f = 0$
- Intuitively we may visualize $A\delta(f)$ as an infinitely tall, infinitely narrow rectangular pulse of **area *A***



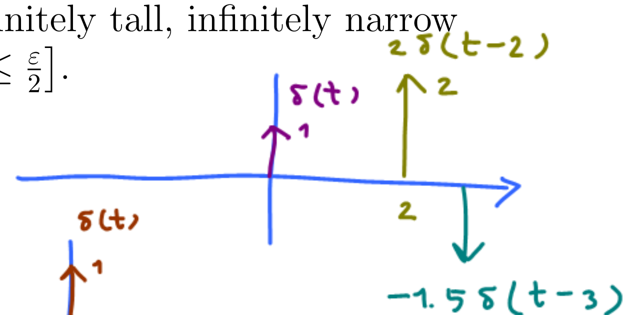
Definition 2.15. The (Dirac) **delta function** or **(unit) impulse function** is denoted by $\delta(t)$. It is usually depicted as a vertical arrow at the origin. Note that $\delta(t)$ is *not*[8] a true function; it is undefined at $t = 0$. We define $\delta(t)$ as a generalized function which satisfies the **sampling property** (or **sifting property**)

$g(t) \equiv 1$

$$\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0) \qquad \int_{-\infty}^{\infty} \delta(t)dt = g(0) = 1 \tag{16}$$

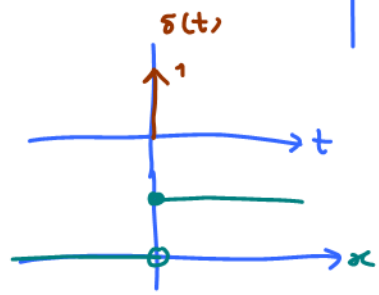
for any function $g(t)$ which is continuous at $t = 0$.

- In this way, the delta "function" has no mathematical or physical meaning unless it appears under the operation of integration.
- Intuitively we may visualize $\delta(t)$ as an infinitely tall, infinitely narrow rectangular pulse of unit area: $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} 1\left[|t| \le \frac{\varepsilon}{2}\right]$.

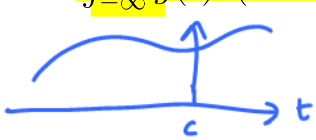


2.16. Properties of $\delta(t)$:

- $\delta(t) = 0$ for $t \neq 0$.
  $\delta(t - T) = 0$ for $t \neq T$.
- $\int_A \delta(t)dt = 1_A(0)$. $= \begin{cases} 1, & 0 \in A \\ 0, & 0 \notin A \end{cases}$



  (a) $\int_{-\infty}^{\infty} \delta(t)dt = 1$.
  (b) $\int_{\{0\}} \delta(t)dt = 1$.
  $\int_{(-\infty,x]} \delta(t)dt =$ (c) $\int_{-\infty}^{x} \delta(t)dt = 1_{[0,\infty)}(x)$. Hence, we may think of $\delta(t)$ as the "derivative" of the **unit step function** $U(t) = 1_{[0,\infty)}(x)$ [13, Defn 3.13 p 126].
- $g(c)\delta(t-c)$
  $\int_{-\infty}^{\infty} g(t)\delta(t - c)dt = g(c)$ for g continuous at c. In fact, for any $\varepsilon > 0$,



$$\int_{c-\varepsilon}^{c+\varepsilon} g(t)\delta(t - c)dt = g(c).$$

- Convolution[9] property:

  $\delta(t) * g(t)$
$$(\delta * g)(t) = (g * \delta)(t) = \int_{-\infty}^{\infty} g(\tau)\delta(t - \tau)d\tau = g(t) \tag{17}$$

  where we assume that g is continuous at t. $= \int_{-\infty}^{\infty} g(t-\tau)\,\delta(\tau)\,d\tau$

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[8]The δ-function is a distribution, not a function. In spite of that, it's always called δ-function.
[9]See Definition 2.37.

sifting property $\;=\; g(t-\tau)\Big|_{\tau=0} = g(t).$

LTI: $\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$

- Factoring a constant a out of the δ-function means scaling it by $\frac{1}{|a|}$:

$$\delta(at) = \frac{1}{|a|}\delta(t). \tag{18}$$

In particular,

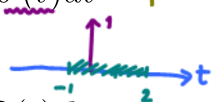
$$\delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f) \tag{19}$$

and

$$\delta(\omega - \omega_0) = \delta(2\pi f - 2\pi f_0) = \frac{1}{2\pi}\delta(f - f_0), \tag{20}$$

where $\omega = 2\pi f$ and $\omega_0 = 2\pi f_0$.

Example 2.17. $\int_{-1}^{2} \delta(t)dt = 1$ and $\int_{1}^{2} \delta(t)dt = 0$.



Example 2.18. $\int_{0}^{2} \delta(t)dt =$

Problem: By writing 0 in $\int_0^2$, do we include 0 in the integration?

Better notation:

$\int_{[0,2]} \delta(t)dt = 1 \qquad \int_{(0,2]} \delta(t)dt = 0$

Example 2.19. $\delta(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1.$

Defn FT

sifting property

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = \left. e^{-j2\pi ft}\right|_{t=0} = e^0 = 1$$

Example 2.20. $\underbrace{e^{j2\pi f_0 t}}_{x(t)} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \underbrace{\delta(f - f_0)}_{X(f)}.$

inverse FT

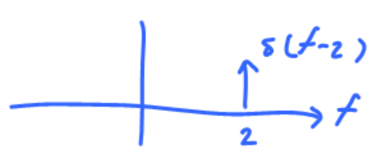
sifting property

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty} \delta(f - f_0)e^{j2\pi ft}df = \left. e^{j2\pi ft}\right|_{f=f_0} = e^{j2\pi f_0 t}$$

Example 2.21. $e^{j\omega_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi\delta(\omega - \omega_0).$

Example 2.22. $e^{j4\pi t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \delta(f-2)$

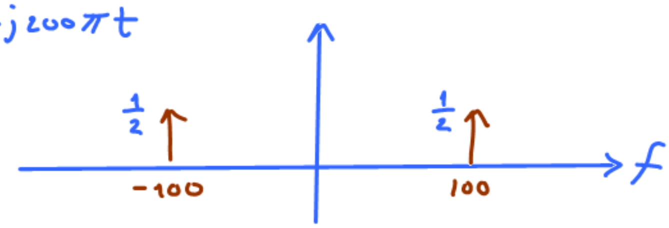
$2\pi(2)t$



$\cos(x) = \frac{1}{2} e^{jx} + \frac{1}{2} e^{-jx}$

$\cos(200\pi t) = \frac{1}{2} e^{j200\pi t} + \frac{1}{2} e^{-j200\pi t}$

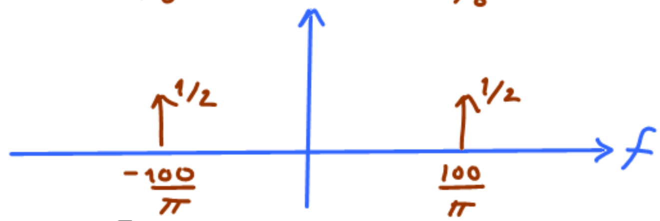
Example 2.23. $\cos(200\pi t) \xrightleftharpoons[\mathcal{F}^{-1}]{\mathcal{F}}$



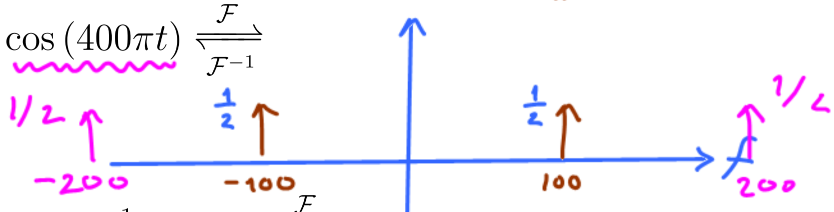
$\cancel{2\pi} f_0 \cancel{t} = \overset{100}{\cancel{200}} \cancel{\pi} \cancel{t}$

2.24. $\cos(2\pi f_0 t) \xrightleftharpoons[\mathcal{F}^{-1}]{\mathcal{F}} \frac{1}{2}\left(\delta\left(f - f_0\right) + \delta\left(f + f_0\right)\right).$

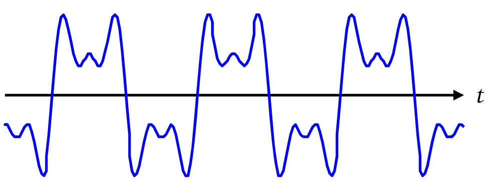
Example 2.25. $\cos(200t) \xrightleftharpoons[\mathcal{F}^{-1}]{\mathcal{F}}$



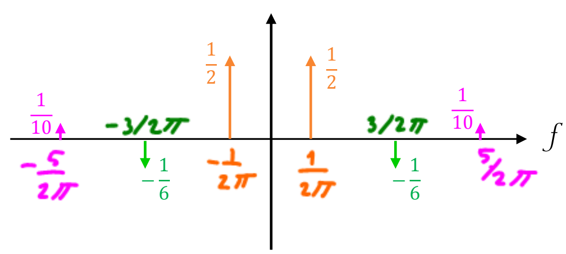
Example 2.26. $\cos\left(200\pi t\right) + \cos\left(400\pi t\right) \xrightleftharpoons[\mathcal{F}^{-1}]{\mathcal{F}}$

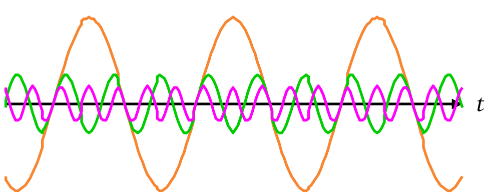


Example 2.27. $\cos\left(t\right) - \frac{1}{3}\cos\left(3t\right) + \frac{1}{5}\cos\left(5t\right) \xrightleftharpoons[\mathcal{F}^{-1}]{\mathcal{F}}$



$g(t) = \cos(t) - \frac{1}{3}\cos(3t) + \frac{1}{5}\cos(5t) \xrightarrow{\mathcal{F}}$





Example 2.28. $\cos^2\left(\overbrace{200\pi t}^{x}\right) \xrightleftharpoons[\mathcal{F}^{-1}]{\mathcal{F}}$

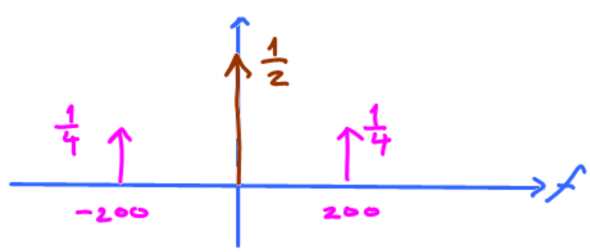
$(\cos x)^2 = \left(\underbrace{\frac{1}{2} e^{jx}}_{A} + \underbrace{\frac{1}{2} e^{-jx}}_{B}\right)^2$

$= A^2 + 2AB + B^2$

$= \frac{1}{4} e^{j2x} + \frac{1}{2} e^{j0} + \frac{1}{4} e^{-j2x}$

$f_0 = 100$ $\qquad$ $f_0 = 200$ $\qquad$ $f_0 = 0$ $\qquad$ $f_0 = -200$

$(\cos(200\pi t))^2 = \frac{1}{4} e^{j400\pi t} + \frac{1}{2} e^{j0} + \frac{1}{4} e^{-j400\pi t}$



Hint: Use problem 1 in HW1

Example 2.29. $\cos(\underbrace{200\pi t}_{x}) \times \cos(\underbrace{400\pi t}_{y}) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}}$

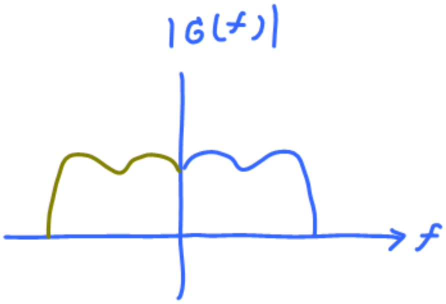
2.30. Conjugate symmetry[10]: If $g(t)$ is **real**-valued, then $G(-f) = (G(f))^*$

$g^*(t) = g(t)$

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$$

$$(G(-f))^* = \int_{-\infty}^{\infty} g^*(t) e^{+j2\pi(-f)t} dt = G(f)$$

$g(t)$



Ex. $G(5) = 4+5j$

$(G(5))^* = 4-5j$

$|G(5)| = \sqrt{4^2+5^2}$

(a) Even amplitude symmetry: $|G(-f)| = |G(f)|$

(b) Odd phase symmetry: $\angle G(-f) = -\angle G(f)$

Observe that if we know $G(f)$ for all f positive, we also know $G(f)$ for all f negative. Interpretation: **Only half of the spectrum contains all of the information.** Positive-frequency part of the spectrum contains all the necessary information. The negative-frequency half of the spectrum can be determined by simply complex conjugating the positive-frequency half of the spectrum.

Furthermore,

(a) If $g(t)$ is real and even, then so is $G(f)$.

(b) If $g(t)$ is real and odd, then $G(f)$ is pure imaginary and odd.

[10]Hermitian symmetry in [3, p 48] and [9, p 17].

2.31. ***Shifting*** properties $g(t) \xrightarrow{\mathcal{F}} G(f)$

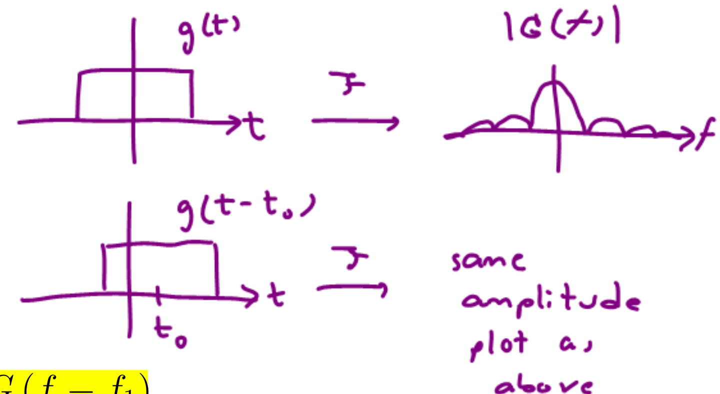
- ***Time-shift***:

$$y(t) \equiv g(t - t_1) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} e^{-j2\pi f t_1} G(f) \equiv Y(f)$$

$|z_1 z_2| = |z_1||z_2|$

  - Note that $|e^{-j2\pi f t_1}| = 1$. So, the spectrum of $g(t - t_1)$ looks exactly the same as the spectrum of $g(t)$ (unless you also look at their phases).

$$|Y(f)| = |e^{-j2\pi f t_1} G(f)| = \underbrace{|e^{-j2\pi f t_1}|}_{} |G(f)| = 1 \; |G(f)| = |G(f)|$$



- ***Frequency-shift*** (or modulation):

$$g(t) = A_0 e^{j2\pi f_0 t}$$

$$e^{j2\pi f_1 t} g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G(f - f_1)$$

$$g(t) e^{j2\pi f_1 t} = A_0 e^{j2\pi f_0 t} e^{j2\pi f_1 t} = A_0 e^{j2\pi (f_0 + f_1) t}$$

2.32. Let $g(t)$, $g_1(t)$, and $g_2(t)$ denote signals with $G(f)$, $G_1(f)$, and $G_2(f)$ denoting their respective Fourier transforms.

(a) ***Superposition theorem*** (linearity): followed directly from linearity of integration

$$a_1 g_1(t) + a_2 g_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} a_1 G_1(f) + a_2 G_2(f).$$

(b) ***Scale-change*** theorem (scaling property [5, p 88]; reciprocal spreading [3, p 46]):

$g(t) \xrightarrow{\mathcal{F}} G(f)$

$$g(at) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{|a|} G\left(\frac{f}{a}\right). \tag{21}$$

- The function $g(at)$ represents the function $g(t)$ *compressed* in time by a factor a (when $|a| > 1$).
- The function $G(f/a)$ represents the function $G(f)$ *expanded* in frequency by the same factor a.

- The scaling property says that
  - if we "squeeze" a function in t, its Fourier transform "stretches out" in f,
  - it is not possible to arbitrarily concentrate a function and its Fourier transform simultaneously,
  - generally speaking, the more concentrated $g(t)$ is, the more spread out its Fourier transform $G(f)$ must be.

  This trade-off can be formalized in the form of an *uncertainty principle*. See also 2.45 and 2.46.

- Intuitively, we understand that compression in time by a factor a means that the signal is varying more rapidly by the same factor. To synthesize such a signal, the frequencies of its sinusoidal components must be increased by the factor a, implying that its frequency spectrum is expanded by the factor a.

  Similarly, a signal expanded in time varies more slowly; hence, the frequencies of its components are lowered, implying that its frequency spectrum is compressed.

(c) ***Duality theorem*** (Symmetry Property [5, p 86]):

suppose we know
a FT pair
$g(t) \xrightarrow{\mathcal{F}} G(f)$
then

$$G(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} g(-f).$$

- In words, for any result or relationship between $g(t)$ and $G(f)$, there exists a dual result or relationship, obtained by interchanging the roles of $g(t)$ and $G(f)$ in the original result (along with some minor modifications arising because of a sign change).

  In particular, if the Fourier transform of $g(t)$ is $G(f)$, then the Fourier transform of $G(f)$ with f replaced by t is the original time-domain signal with t replaced by $-f$.

- If we use the ω-definition (9), we get a similar relationship with an extra factor of 2π:

$$\hat{G}(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi g(-\omega).$$

Example 2.33. Let's try to use the scale-change theorem to double-check the Fourier transform of a simple function. Consider the function $x(t) = g(at)$ where

$$g(t) = e^{j2\pi f_0 t}.$$

Note that $g(t)$ is simply a complex exponential function at frequency f_0. From Example 2.20, its Fourier transform $G(f)$ is simply $\delta(f - f_0)$.

(a) From $x(t) = g(at) = e^{j2\pi f_0 (at)}$, by grouping the factor a with f_0, we get

$$x(t) = e^{j2\pi (af_0) t}.$$

Therefore, $x(t)$ is a complex exponential function at frequency af_0. As in Example 2.20, its Fourier transform is

$$X(f) = \delta(f - af_0).$$

(b) Alternatively, we can also apply the scale-change theorem. From $x(t) = g(at)$, we know that $X(f) = \frac{1}{|a|} G\left(\frac{f}{a}\right)$. Plugging in $G(f) = \delta(f - f_0)$, we get

$$X(f) = \frac{1}{|a|}\delta\left(\frac{f}{a} - f_0\right) = \frac{1}{|a|}\delta\left(\frac{1}{a}(f - af_0)\right).$$

Now, recall, from 2.16 that, factoring a constant α out of the δ-function means scaling it by $\frac{1}{|\alpha|}$. Here, the constant is $\alpha = \frac{1}{a}$. Therefore,

$$X(f) = \frac{1}{|a|}\frac{1}{\left|\frac{1}{a}\right|}\delta\left(f - af_0\right) = \delta\left(f - af_0\right).$$

Exercise 2.34. Similar to Example 2.33, one can also try to apply the scale-change theorem to show that

$$x(t) = \cos(2\pi a f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(\delta(f - af_0) + \delta(f + af_0)\right).$$

Example 2.35. From Example 2.13, we know that

$$1\left[|t| \le a\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2a \ \operatorname{sinc}\left(2\pi a f\right) \tag{22}$$

By the duality theorem, we have

$$2a \operatorname{sinc}(2\pi a t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1[|-f| \le a],$$

which is the same as

$$\operatorname{sinc}(2\pi f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2f_0} 1[|f| \le f_0]. \tag{23}$$

Both transform pairs are illustrated in Figure 6.

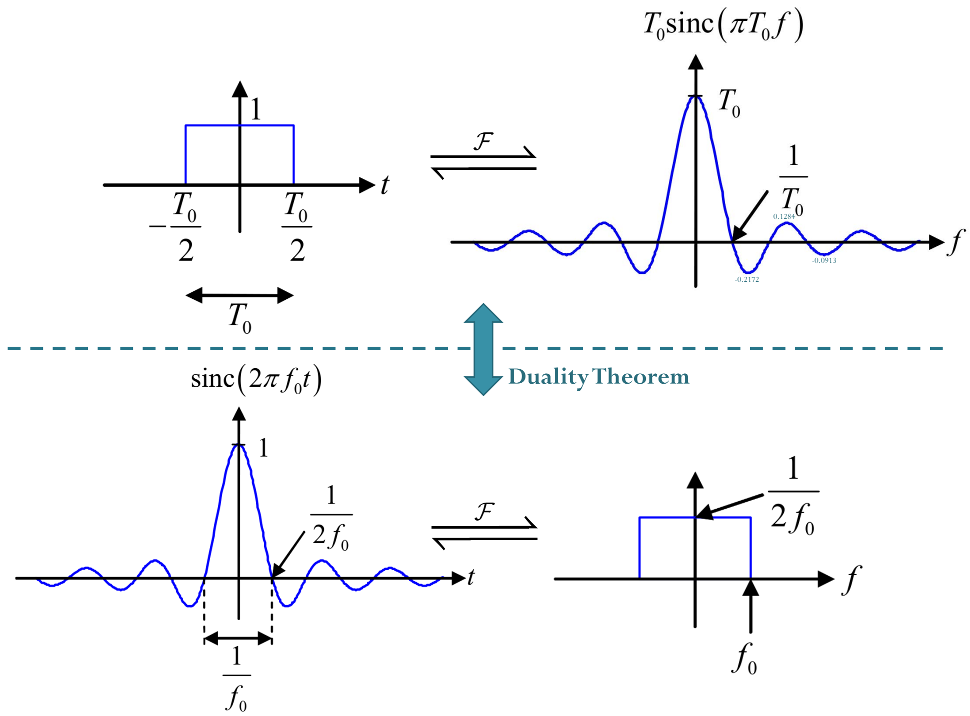


Figure 6: Duality theorem: rectangular and sinc functions

Example 2.36. Let's try to derive the time-shift property from the frequency-shift property. We start with an arbitrary function $g(t)$. Next we will define another function $x(t)$ by setting $X(f)$ to be $g(f)$. Note that f here is just a dummy variable; we can also write $X(t) = g(t)$. Applying the duality theorem to the transform pair $x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X(f)$, we get another transform pair $X(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} x(-f)$. The LHS is $g(t)$; therefore, the RHS must be $G(f)$. This implies $G(f) = x(-f)$. Next, recall the frequency-shift property:

$$e^{j2\pi ct} x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} X(f-c).$$

The duality theorem then gives

$$X(t-c) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} e^{j2\pi c-f} x(-f).$$

Replacing $X(t)$ by $g(t)$ and $x(-f)$ by $G(f)$, we finally get the time-shift property.

Definition 2.37. The **convolution** of two signals, $g_1(t)$ and $g_2(t)$, is a new function of time, $g(t)$. We write

$$g = g_1 * g_2.$$

It is defined as the integral of the product of the two functions after one is reversed and shifted:

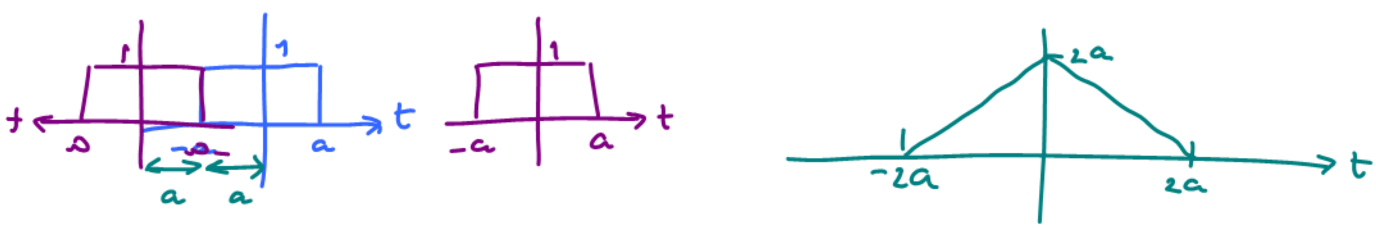
$$g(t) = (g_1 * g_2)(t) \tag{24}$$

$$= \int_{-\infty}^{+\infty} g_1(\mu) g_2(t-\mu) d\mu = \int_{-\infty}^{+\infty} g_1(t-\mu) g_2(\mu) d\mu. \tag{25}$$

- Note that t is a parameter as far as the integration is concerned.
- The integrand is formed from g_1 and g_2 by three operations:
  (a) time reversal to obtain $g_2(-\mu)$,
  (b) time shifting to obtain $g_2(-(\mu - t)) = g_2(t - \mu)$, and
  (c) multiplication of $g_1(\mu)$ and $g_2(t-\mu)$ to form the integrand.
- In some references, (24) is expressed as $g(t) = g_1(t) * g_2(t)$.

Example 2.38. We can get a triangle from convolution of two rectangular waves. In particular,

$$1[|t| \le a] * 1[|t| \le a] = (2a - |t|) \times 1[|t| \le 2a].$$



2.39. Convolution properties involving the δ-function:

$\delta(t) * g(t) = g(t)$ $\qquad$ $\delta(t-a) * g(t) = g(t-a)$

$g(t) \longrightarrow \boxed{\delta(t)} \longrightarrow g(t)$ $\qquad$ $g(t) \longrightarrow \boxed{\delta(t-a)} \longrightarrow g(t-a)$

$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$

2.40. Convolution theorems:

(a) Convolution-in-time rule:

$$g_1 * g_2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_1 \times G_2. \tag{26}$$

(b) Convolution-in-frequency rule:

$$g_1 \times g_2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_1 * G_2. \tag{27}$$

Example 2.41. We can use the convolution theorem to "prove" the frequency-shift property in 2.31.

$$e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f-f_0)$$

$$e^{j2\pi f_0 t} g(t) = e^{j2\pi f_0 t} \times g(t) \xrightarrow{\mathcal{F}} \delta(f-f_0) * G(f) = G(f-f_0)$$

2.42. From the convolution theorem, we have

- $g^2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G * G$
- if g is band-limited to B, then g^2 is band-limited to $2B$

2.43. Parseval's theorem (Rayleigh's energy theorem, Plancherel formula) for Fourier transform:

$$E_g \equiv \int_{-\infty}^{+\infty} |g(t)|^2 dt = \int_{-\infty}^{+\infty} |G(f)|^2 df. \tag{28}$$

The LHS of (28) is called the (total) **energy** of $g(t)$. On the RHS, $|G(f)|^2$ is called the energy spectral density of $g(t)$. By integrating the energy spectral density over all frequency, we obtain the signal 's total energy. The energy contained in the frequency band B can be found from the integral $\int_B |G(f)|^2 df$.

More generally, Fourier transform preserves the inner product [2, Theorem 2.12]:

$$\langle g_1, g_2 \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt = \int_{-\infty}^{\infty} G_1(f) G_2^*(f) df = \langle G_1, G_2 \rangle.$$

Example 2.44. Perform the following integration graphically with the help of Fourier transform properties:

(a) $\int\limits_{-\infty}^{\infty} \underbrace{\text{sinc}(t)}_{g(t)}\, dt.$

$g(t) = \text{sinc}(t) = \text{sinc}\left(2\pi\left(\frac{1}{2\pi}\right)t\right)$

"freq" of sinc $= \frac{1}{2\pi}$

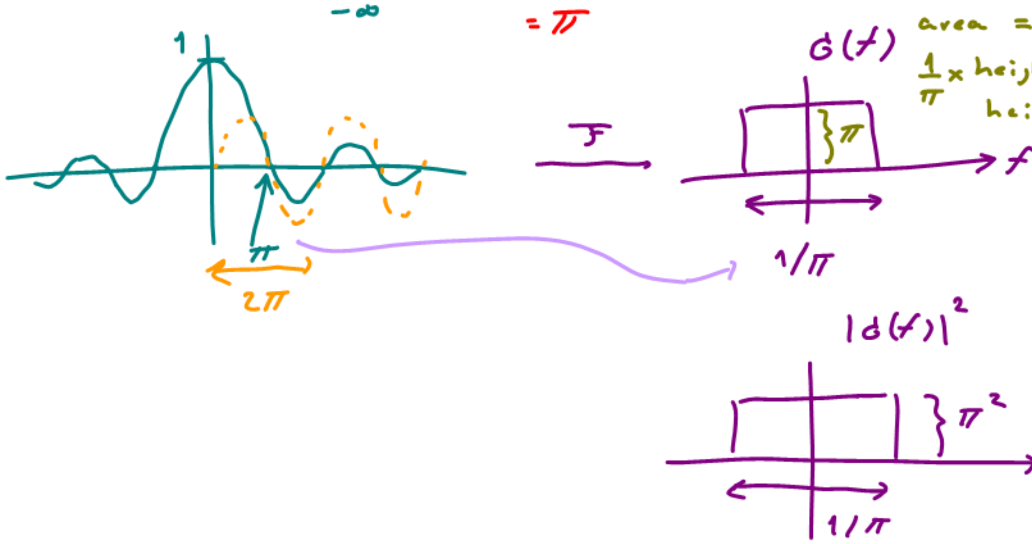
"period" $= 2\pi$

We know that $\int\limits_{-\infty}^{\infty} g(t)\,dt = G(f)\Big|_{f=0}$ $= \pi$

area $= 1$

$\frac{1}{\pi} \times$ height $= 1$

height $= \pi$



(b) $\int\limits_{-\infty}^{\infty} \text{sinc}^2(t)\, dt.$ $= \int\limits_{-\infty}^{\infty} g^2(t)\,dt = \int\limits_{-\infty}^{\infty} |G(f)|^2\,df = \pi^2 \times \frac{1}{\pi} = \pi$

Parseval's theorem

2.45. (Heisenberg) **Uncertainty Principle** [2, 11]: Suppose g is a function which satisfies the normalizing condition $\|g\|_2^2 = \int |g(t)|^2 dt = 1$ which automatically implies that $\|G\|_2^2 = \int |G(f)|^2 df = 1$. Then

$$\left(\int t^2 |g(t)|^2 dt\right)\left(\int f^2 |G(f)|^2 df\right) \geq \frac{1}{16\pi^2}, \tag{29}$$

and equality holds if and only if $g(t) = Ae^{-Bt^2}$ where $B > 0$ and $|A|^2 = \sqrt{2B/\pi}$.

- In fact, we have
$$\left(\int t^2 |g(t-t_0)|^2 dt\right)\left(\int f^2 |G(f-f_0)|^2 df\right) \geq \frac{1}{16\pi^2},$$
for every t_0, f_0.
- The proof relies on Cauchy-Schwarz inequality.
- For any function h, define its dispersion Δ_h as $\frac{\int t^2 |h(t)|^2 dt}{\int |h(t)|^2 dt}$. Then, we can apply (29) to the function $g(t) = h(t)/\|h\|_2$ and get
$$\Delta_h \times \Delta_H \geq \frac{1}{16\pi^2}.$$

We can find a positive constant T_0 such that $g(t) = 0$ for $|t| > T_0$

2.46. A signal cannot be simultaneously time-limited and band-limited.

Proof. Suppose $g(t)$ is simultaneously (1) time-limited to T_0 and (2) band-limited to B. Pick any positive number T_s and positive integer K such that $f_s = \frac{1}{T_s} > 2B$ and $K > \frac{T_0}{T_s}$. The sampled signal $g_{T_s}(t)$ is given by

We can find a positive constant B such that $G(f) = 0$ for $|f| > B$

$$g_{T_s}(t) = \sum_k g[k]\delta(t - kT_s) = \sum_{k=-K}^{K} g[k]\delta(t - kT_s)$$

where $g[k] = g(kT_s)$. Now, because we sample the signal faster than the Nyquist rate, we can reconstruct the signal g by producing $g_{T_s} * h_r$ where the LPF h_r is given by

$$H_r(\omega) = T_s 1[\omega < 2\pi f_c]$$

with the restriction that $B < f_c < \frac{1}{T_s} - B$. In frequency domain, we have

$$G(\omega) = \sum_{k=-K}^{K} g[k] e^{-jk\omega T_s} H_r(\omega).$$

Consider ω inside the interval $I = (2\pi B, 2\pi f_c)$. Then,

$$0 \overset{\omega > 2\pi B}{=} G(\omega) \overset{\omega < 2\pi f_c}{=} T_s \sum_{k=-K}^{K} g\left(kT_s\right) e^{-jk\omega T_s} \overset{z = e^{j\omega T_s}}{=} T_s \sum_{k=-K}^{K} g\left(kT_s\right) z^{-k} \tag{30}$$

Because $z \neq 0$, we can divide (30) by z^{-K} and then the last term becomes a polynomial of the form

$$a_{2K} z^{2K} + a_{2K-1} z^{2K-1} + \cdots + a_1 z + a_0.$$

By fundamental theorem of algebra, this polynomial has only finitely many roots– that is there are only finitely many values of $z = e^{j\omega T_s}$ which satisfies (30). Because there are uncountably many values of ω in the interval I and hence uncountably many values of $z = e^{j\omega T_s}$ which satisfy (30), we have a contradiction. $\square$

2.47. The observation in 2.46 raises concerns about the signal and filter models used in the study of communication systems. Since a signal cannot be both bandlimited and timelimited, we should either abandon bandlimited signals (and ideal filters) or else accept signal models that exist for all time. On the one hand, we recognize that any real signal is timelimited, having starting and ending times. On the other hand, the concepts of bandlimited spectra and ideal filters are too useful and appealing to be dismissed entirely.

The resolution of our dilemma is really not so difficult, requiring but a small compromise. Although a strictly timelimited signal is not strictly bandlimited, its spectrum may be negligibly small above some upper frequency limit B. Likewise, a strictly bandlimited signal may be negligibly small outside a certain time interval $t_1 \le t \le t_2$. Therefore, we will often assume that signals are essentially both bandlimited and timelimited for most practical purposes.